Exercise Sheet 2

1) Let $A, B$ be events and $0 < P(B) < 1$. Prove that the following statements are equivalent.
   i) $A$ is independent of $B$.
   ii) $P(A) \cdot P(B) = P(A \cap B)$.
   iii) $P(A|B) = P(A)$.

2) Let $(\Omega, 2^\Omega, P)$ be a discrete probability space with $\Omega$ being finite and $X : \Omega \to \mathbb{R}$. Prove that, for $A \subset X(\Omega)$, the value
   \[ P_X(A) := P(X^{-1}(A)) = P(\{\omega \in \Omega \mid X(\omega) \in A\}) \]
   is a probability. (Hint: Note that if $\Omega$ is finite, the second axiom of Kolmogorov’s probability definition reduces to “If $A$ and $B$ are disjoint subsets of $\Omega$ then $P(A \cup B) = P(A) + P(B)$”.)

3) Consider a game of chance where a player bets one euro and picks a number between 1 and 6, e.g. 2. A die is then rolled three times and the player receives
   - nothing if no 2 was thrown,
   - two euros if 2 was thrown one time,
   - three euros if 2 was thrown two times,
   - four euros if 2 was thrown three times,

and similar for the other numbers. Define a random variable $Z$ that describes the net profit (“Reingewinn”) of one round of the game. Calculate the probabilities $P(Z = -1), P(Z = 0), P(Z = 1), P(Z = 2), P(Z = 3)$, as well as the expectation, variance, and standard deviation of $Z$.

4. Let $X, Y$ be discrete real-valued random variables for which the expectations exist. Show the following properties:
   - $E(X + Y) = E(X) + E(Y)$.
   - If $X$ and $Y$ are independent, then $E(X \cdot Y) = E(X) \cdot E(Y)$.

5. A Poisson distributed random variable $X$ has the following probability distribution:
   \[ P(X = k) = \frac{\mu^k}{k!} e^{-\mu} \quad k \in \{0, 1, \ldots\} \]
   Prove that mean and variance of $X$ is equal to $\mu$. (Hint: $e^a = \sum_{i=0}^{\infty} \frac{a^i}{i!}$.)